Introduction to Dynamical Systems

Solutions Problem Set 9

Exercise 1. Prove Lemma 3.2 in Lecture9.pdf

Solution. To show this we can simply use the Jordan form of the matrix. Recall the statement of the lemma:

Lemma

If the matrix A is hyperbolic in the ODE system sense, so that e^A is hyperbolic in the sense of Lecture8.pdf, then there is a splitting

$$\mathbb{R}^n = E_+ \oplus E_-$$

as well as constants $c > 0$ and $\theta_\pm \in (0,1)$ such that we have $e^{tA}\big|_{E_\pm} : E_\pm \longrightarrow E_\pm$, and further, we have

$$\begin{aligned} \left|e^{tA}y_0\right| &\leq c \cdot \theta_+^t \cdot |y_0|, \quad \forall y_0 \in E_+, \quad t \geq 0 \\ \left|e^{-tA}y_0\right| &\leq c \cdot \theta_-^t \cdot |y_0|, \quad \forall y_0 \in E_-, \quad t \geq 0 \end{aligned}$$

We begin by noting that we have $e^{tA} = P^{-1}e^{tJ}P$, where J is the Jordan normal form of A. We may express each block J_λ in J as $J_\lambda = \lambda I + N$, where either

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

or $N = 0$ is a nilpotent matrix (i.e. $N^q = 0$ for some $q \geq 0$), which then results in

$$e^{tJ_\lambda} = e^{t\lambda I}e^{tN} = e^{t\lambda}I + e^{t\lambda}\sum_{j=1}^{q}\frac{N^j}{j!}t^j = e^{t\lambda}I + e^{t\lambda}\tilde{N}.$$

Upon noticing that N^j shifts the diagonal above the main diagonal one position above, we conclude that the sum results in a matrix $\tilde{N}$ with polynomial entries with degree at most $q-1$. The factor $e^{t\lambda}$, however, imposes the decay of the entries as $t \to \infty$. Letting $n \in \mathbb{N}$ and Λ be the largest eigenvalue of A, we set $\sigma_+ = e^\Lambda + 1/m$, with m large enough such that $\sigma_+ \in (e^\Lambda, 1)$. Then, for $t \geq T_m$ large, and letting L be the set of eigenvalues of A (with multiplicity), we have

$$\begin{aligned} \left|e^{tA}x\right| &= \left|\sum_{\lambda \in L} e^{t\lambda}Ix + e^{t\lambda}\tilde{N}_\lambda\right| x \leq \sum_{\lambda \in L}\left|e^{t\lambda}Ix\right| + \left|e^{t\lambda}\tilde{N}_\lambda x\right| \\ &\leq e^{t\Lambda}|x| + \sum_{\lambda \in L}\left\|e^{t\lambda}\tilde{N}_\lambda\right\||x| \leq e^{t\Lambda}|x| + \frac{1}{m}|x| = \theta_+|x|, \end{aligned}$$

since $\left\|e^{t\lambda}\tilde{N}\right\| \to 0$ as $t \to \infty$. Then, set $c_+ = \max_{[0,T_m]}\left\|e^{tA}\right\|/\theta_+^t$ (which is finite since that function is continuous on the compact interval $[0, T_m]$) to conclude. ☐

Exercise 2. Give a careful proof of Lemma 3.3 in Lecture9.pdf.

Solution. Follow the sketch in page 4 and carefully compute the C^1-norm

$$\|g\|_{C^1} = \sup_x |g(x)| + \sup_x |g'(x)|$$

of the difference $Ty_1 - Ty_2$, given by

$$Ty(t) = \int_0^t e^{(t-s)Df(0)} F(y(s))\,\mathrm{d}s$$

to show it is a contraction. □

Exercise 3. Using the result from **Exercise 2** in Problem Set 8, prove that if $0 \in \mathbb{R}^n$ is a hyperbolic fixed point of the system of ODEs

$$\dot{y}_j = f_j(y_1, \ldots, y_n), \quad j = 1, 2, \ldots, n, \tag{1}$$

and more specifically if $Df(0)$ *only has eigenvalues with negative real part* then there is a neighborhood $U \ni 0$ in $\mathbb{R}^n$ such that the solution to the system (1) with initial condition $y(0) =\in U$ satisfies a bound of the form

$$|y(t)| \leq Ce^{-ct}, \quad t \geq 0$$

for some suitable constants $C, c > 0$.

Solution. We may apply **Exercise 2** in Problem Set 8 to both h and h^{-1} (simply by switching the roles of $\hat{\psi}$ and $\hat{\phi}$) in the statement of the Hartman-Grobman theorem in order to find that both functions are Hölder continuous (at least locally uniformly) and that $y(t) = h^{-1}\left(e^{Df(0)t}h(y(0))\right)$, and estimate -using **Exercise 1**- as follows

$$|y(t)| \leq [h^{-1}]_\alpha \left|e^{Df(0)t}h(y(0))\right|^\alpha \leq [h^{-1}]_\alpha \left(K[h]_\alpha e^{-kt} |y(0)|^\alpha\right)^\alpha \leq Ce^{-ct}.$$

□